A New Application of Laplace Decomposition Algorithm for Handling Linear Volterra Integral Equations

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***Abstract*- In this paper, the Laplace Decomposition Algorithm (LDA) is employed to obtain approximate analytical solution of linear Volterra integral equation. The result show that the method converges rapidly and approximates the exact solution very accurately using only few iterates of the recursive scheme.**

***Keywords*- Linear Volterra integral equation, Laplace transform, Convolution theorem, Inverse Laplace transform, Laplace decomposition algorithm, Noise terms phenomenon.**

I. INTRODUCTION

Volterra examined the linear Volterra integral equation of the form [1-5]

$$u(x)=f(x)+\lambda \int_0^x K(x,t)\,u(t)dt \qquad (1)$$

where the unknown function u(x), that will be determined, occurs inside and outside the integral sign. The kernel K(x,t) and the function f(x) are given real-valued functions, and λ is a parameter. The Volterra integral equations appear in many physical applications such as neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating.

The Laplace decomposition algorithm was first proposed by Khuri[6-7]. Wazwaz[8] established a necessary condition that is essentially needed to ensure the appearance of "noise terms" in the inhomogeneous equations. The necessary condition for the "noise terms" to appear in the components u0 and u1 is that the exact solution must appear as the part of u0 among other terms. Khan and Gondal [9] applied Laplace decomposition method for a new analytical solution of foam drainage equation.

The restriction and improvements of Laplace decomposition method was given by Khan and Gondal[10]. Zafar et.al. used Laplace decomposition method to solve Burger's equaton[11]. It is worth mentioning that the proposed method is an elegant combination of Laplace transform and decomposition algorithm. The advantage of this proposed method is its capability of combining two powerful methods for obtaining exact solution. The aim of this work is to establish exact solutions or approximate solutions of high degree of accuracy for the linear Volterra integral equations without large computational work.

II. THE LAPLACE DECOMPOSITION ALGORITHM

In this work we will assume that the kernel K(x,t) of (1) is a difference kernel that can be expressed by the difference (x-t). The linear Volterra integral equation (1) can thus be expressed as

$$u(x)=f(x)+\lambda \int_0^x K(x-t)\,u(t)dt \qquad (2)$$

Applying the Laplace transform to both sides of (2), we have,

$$L\{u(x)\}=L\{f(x)\}+\lambda L\{\int_0^x K(x-t)\,u(t)dt\} \qquad (3)$$

Using Convolution theorem of the Laplace transform, we have

$$L\{u(x)\}=L\{f(x)\}+\lambda L\{K(x)\}\,L\{u(x)\} \qquad (4)$$

Operating inverse Laplace transform on both sides of (4), we have

$$u(x)=f(x)+\lambda L^{-1}\{L\{K(x)\}\,L\{u(x)\}\} \qquad (5)$$

The Laplace decomposition algorithm (LDA) assumes the solution u(x) can be expanded into infinite series as

$$u(x)=\sum_{n=0}^{\infty} u_n(x) \qquad (6)$$

By substituting (6) in (5), the solution can be written as:

$$\sum_{n=0}^{\infty} u_n(x)=f(x)+\lambda L^{-1}\{L\{K(x)\}L\{\sum_{n=0}^{\infty} u_n(x)\}\} \qquad (7)$$

In general, the recursive relation is given by:

$$u_0(x)=f(x),$$

$$u_{n+1}(x)=\lambda L^{-1}\{L\{K(x)\}L\{\sum_{n=0}^{\infty} u_n(x)\}\},\ n\geq 0 \qquad (8)$$

III. APPLICATIONS

In this section, some applications are given in order to demonstrate the effectiveness of Laplace decomposition algorithm (LDA) for linear Volterra integral equation.

A. APPLICATION:1

Consider linear Volterra integral equation with $\lambda=-1$

$u(x)=x+x^2+x^4/12+\sin x+2\cos x-2-$

$\int_0^x (x-t)^2 u(t)dt$ (9)

Applying the Laplace transform to both sides of (9), we have,

$L\{u(x)\} = 1/s^2+2/s^3+2/s^5+1/(s^2+1)+2s/(s^2+1)-2/s-L\{\int_0^x (x-t)^2 u(t)dt\}$ (10)

Using Convolution theorem of the Laplace transform, we have

$L\{u(x)\} = 1/s^2+2/s^3+2/s^5+1/(s^2+1)+2s/(s^2+1)-2/s-2/s^3 L\{u(x)\}$ (11)

Operating inverse Laplace transform on both sides of (11), we have

$u(x)= x+x^2+x^4/12+\sin x+2\cos x-2-L^{-1}\{2/s^3 L\{u(x)\}\}$ (12)

The Laplace decomposition algorithm (LDA) assumes the solution u can be expanded into infinite series as

$\sum_{n=0}^{\infty} u=u_n$ (13)

By substituting (13) in (12), the solution can be written as:

$\sum_{n=0}^{\infty} u_n(x)=x+x^2+x^4/12+\sin x+2\cos x-2-L^{-1}\{2/s^3 L\{\sum_{n=0}^{\infty} u_n(x)\}\}$ (14)

From (14) our required recursive relation is given by:

$u_0(x)=x+x^2+x^4/12+\sin x+2\cos x-2$ (15)

$u_{n+1}(x)=-L^{-1}\{2/s^3 L\{u_n(x)\}\},\ n\geq 0$ (16)

The first few components of un(x) by using recursive relation (16) as follows immediately

$u_1(x)=2-4x-x^2+2x^3/3-x^4/12-x^5/30-x^7/1260+4\sin x-2\cos x$ (17)

The noise terms appear in $u_0(x)$ and $u_1(x)$. Therefore, by noise terms phenomena, we have

$u(x)=x+\sin x$, which is the exact solution and satisfies the integral equation (9).

B. APPLICATION: 2

Consider linear Volterra integral equation with $\lambda=-1$

$u(x)=1-\int_0^x (x-t)u(t)dt$ (18)

Applying the Laplace transform to both sides of (18), we have,

$L\{u(x)\}=1/s-L\{\int_0^x (x-t)u(t)dt\}$ (19)

Using Convolution theorem of the Laplace transform, we have

$L\{u(x)\}=1/s-1/s^2 L\{u(x)\}$ (20)

Operating inverse Laplace transform on both sides of (20), we have

$u(x)=1-L^{-1}\{1/s^2 L\{u(x)\}\}$ (21)

The Laplace decomposition algorithm (LDA) assumes the solution u can be expanded into infinite series as

$u=\sum_{n=0}^{\infty} u_n$ (22)

By substituting (22) in (21), the solution can be written as:

$\sum_{n=0}^{\infty} u_n(x)=1-L^{-1}\{1/s^2 L\{\sum_{n=0}^{\infty} u_n(x)\}\}$ (23)

From (23) our required recursive relation is given by:

$u_0(x)=1$ (24)

$u_{n+1}(x)=-L^{-1}\{1/s^2 L\{un(x)\}\},\ n\geq 0.$ (25)

The first few components of $u_n(x)$ by using recursive relation (25) as follows immediately

$u_1(x)=-x^2/2!$ (26)

$u_2(x)=x^4/4!$ (27)

and so on. Using (22), the series solution is therefore given by

$u(x)=1-x^2/2!+x^4/4!-\ldots\ldots\ldots\ldots\ldots$ (28)

that converges to the exact solution

$u(x)=\cos x.$ (29)

IV. CONCLUSION

In this paper, we have successfully developed the Laplace decomposition algorithm (LDA) for linear Volterra Integral equation. The given applications showed that the exact solution have been obtained even with just the two first terms of the LDA solution, which indicates that the proposed method LDA need much less computational work. The proposed scheme can be applied for other linear Volterra integral equations.

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