Solution of Linear Volterra Integro-Differential Equations of Second Kind by Using Laplace Decomposition Algorithm

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Abstract: **In this paper, Laplace decomposition algorithm is introduced for the approximate analytical solution of linear Volterra integro-differential equations of second kind. The technique is described and illustrated with some numerical applications. The results assert that this scheme is rapidly convergent and give the exact result using only few terms of its iteration scheme.**

Keywords: **Laplace decomposition algorithm, linear Volterra integro-differential equations, Laplace transform, inverse Laplace transform, convolution theorem.**

I INTRODUCTION

Mathematical modeling of real life problems usually results in functional equations e.g. partial differential equations, integral and integro-differential equations, stochastic equations and others. In particular integro-differential equations arise in many scientific and engineering applications such as glass forming process, heat transfer, diffusion process, in general neutron diffusion, nano-hydrodynamics and biological species coexisting together with increasing and decreasing rates of generating and wind ripple in the desert.

Volterra studied the hereditary influences when he was examining a population growth model. The research work resulted in a specific topic, where both differential and integral operators appeared together in the same equation. This new type of equations was termed as Volterra integro-differential equations [1-5] given in the form

$$v^{n}(x) = f(x) + \lambda \int_{0}^{x} k(x,t)v(t)dt \qquad (1)$$

where $v^{n}(x) = \frac{d^{n}v}{dx^{n}}$. Because the resulted equation (1) combines the differential and integral operators, then it is necessary to define initial conditions $v(0), v'(0), \ldots\ldots, v^{(n-1)}(0)$ for the determination of the particular solution v(x) of the Volterra integro-differential equation (1).

Any Volterra integro-differential equation is characterized by the existence of one or more of the derivatives $v'(x), v''(x), v'''(x), \ldots$ outside the integral sign. Volterra integro-differential equations may be observed when we convert an initial value problem to an integral equation by using Leibnitz rule. In this work, we used Laplace decomposition algorithm because this scheme provides the solution in a rapidly convergent series with components that are elegantly computed. The Laplace decomposition algorithm was first proposed by Khuri [6-7] which is further used by Yusufoglu [8] to solve duffing equation and Elgazery [9] for Flalkner-skan equation. The modification of Laplace decomposition method introduced by Hussain and Khan [10]. Zafar et. al. [11] used Laplace decomposition method to solve Burger's equation. Khan and Gondal [12] applied Laplace decomposition method for a new analytical solution of foam drainage equation. Khan and Hussain [13] applied Laplace decomposition method on semi-infinite domain. The restrictions and improvements of Laplace decomposition method was given by Khan and Gondal[14]. Sudhanshu et. al. [15] applied Laplace decomposition algorithm to solve the system of homogeneous linear partial differential equations. Sudhanshu et. al. [16] used Laplace decomposition algorithm to solve the system of weakly singular Volterra integral equations. A new application of Laplace decomposition algorithm for handling linear Volterra integral equations was given by Sudhanshu et. al. [17].

It is worth mentioning that the proposed method is an elegant combination of Laplace transform and decomposition algorithm. The advantage of this proposed method is its capability of combining two powerful methods for obtaining exact solution. The aim of this work is to establish exact solutions or approximate solutions of high degree of accuracy for Linear Volterra integro-differential equations of second kind.

II LAPLACE DECOMPOSITION ALGORITHM

In this section, we present Laplace decomposition algorithm for solving Linear Volterra integro-differential equations of second kind given by (1). In this work, we will assume that the kernel k(x,t) of (1) is a difference

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kernel that can be expressed by difference (x-t). The linear Volterra integro-differential equation of second kind (1) can thus be expressed as

$$v^{n}(x) = f(x) + \lambda \int_0^x k(x-t)v(t)dt \quad (2)$$

With

$$v(0) = a_0, v'(0) = a_1, \dots, v^{(n-1)}(0) = a_{n-1} \quad (3)$$

Applying the Laplace transform to both sides of (2) and using (3), we get

$$p^n L\{v(x)\} = p^{n-1}a_0 - p^{n-2}a_1 \dots \dots - a_{n-1} + L\{f(x)\} + \lambda L\{\int_0^X k(x-t)v(t)dt\} \quad (4)$$

Using convolution theorem of the Laplace transform, we have

$$L\{v(x)\} = \frac{a_0}{p} + \frac{a_1}{p^2} + \cdots \dots \dots + \frac{a_{n-1}}{p^n} + L\{f(x)\} + \lambda L\{k(x)\}L\{v(x)\} \quad (5)$$

Operating inverse Laplace transform on both sides of (5), we have

$$v(x) = a_0 + \frac{a_1 x}{1!} + \cdots \dots \dots + a_{n-1}\frac{x^{n-1}}{(n-1)!} + f(x) + \lambda L^{-1}\{L\{k(x)\}L\{v(x)\}\} \quad (6)$$

The Laplace decomposition algorithm assumes the solution v(x) can be expanded into infinite series as

$$v(x) = \sum_{n=0}^{\infty} v_n(x) \quad (7)$$

By substituting (7) in (6), the solution can be written as

$$\sum_{n=0}^{\infty} v_n(x) = a_0 + \frac{a_1 x}{1!} + \cdots \dots \dots + a_{n-1}\frac{x^{n-1}}{(n-1)!} + f(x) + \lambda L^{-1}\left\{L\{k(x)\}L\{\sum_{n=0}^{\infty} v_n(x)\}\right\} \quad (8)$$

In general, the recursive relation is given by

$$v_0(x) = a_0 + \frac{a_1 x}{1!} + \cdots \dots \dots + a_{n-1}\frac{x^{n-1}}{(n-1)!} + f(x) \quad (9)$$

$$v_{n+1}(x) = \lambda L^{-1}\left\{L\{k(x)\}L\{\sum_{n=0}^{\infty} v_n(x)\}\right\}, \quad n \geq 0 \quad (10)$$

III APPLICATIONS

In this section, some applications are given in order to demonstrate the effectiveness of Laplace decomposition algorithm for Linear Volterra integro-differential equations of second kind.

A. APPLICATION:1

Consider Linear Volterra integro-differential equations of second kind

$$v'(x) = 2 + x - \frac{x^3}{3!} + \int_0^x (x-t)v(t)dt \text{ with } v(0) = 1 \quad (11)$$

Applying the Laplace transform to both sides of (11) and using initial condition, we get

$$L\{v(x)\} = \frac{1}{p} + \frac{2}{p^2} + \frac{1}{p^3} - \frac{1}{p^5} + \frac{1}{p}L\{\int_0^x (x-t)v(t)dt\} \quad (12)$$

Using convolution theorem of the Laplace transform, we have

$$L\{v(x)\} = \frac{1}{p} + \frac{2}{p^2} + \frac{1}{p^3} - \frac{1}{p^5} + \frac{1}{p^3}L\{v(x)\} \quad (13)$$

Operating inverse Laplace transform on both sides of (13), we have

$$v(x) = 1 + 2x + \frac{x^2}{2!} - \frac{x^4}{4!} + L^{-1}\left\{\frac{1}{p^3}L\{v(x)\}\right\} \quad (14)$$

The Laplace decomposition algorithm assumes the solution v(x) can be expanded into infinite series as

$$v(x) = \sum_{n=0}^{\infty} v_n(x) \quad (15)$$

By substituting (15) in (14), the solution can be written as

$$\sum_{n=0}^{\infty} v_n(x) = 1 + 2x + \frac{x^2}{2!} - \frac{x^4}{4!} + L^{-1}\left\{\frac{1}{p^3}L\left\{\sum_{n=0}^{\infty} v_n(x)\right\}\right\} \quad (16)$$

From (16), our required recursive relations are given by

$$v_0(x) = 1 + 2x + \frac{x^2}{2!} - \frac{x^4}{4!} \quad (17)$$

$$v_{n+1}(x) = L^{-1}\left\{\frac{1}{p^3}L\{v_n(x)\}\right\}, n \geq 0 \quad (18)$$

The first few components of $v_n(x)$ by using recursive relation (18) as follows immediately

$$v_1(x) = \frac{x^3}{3!} + \frac{2x^4}{4!} + \frac{x^5}{5!} - \frac{x^7}{7!} \quad (19)$$

$$v_2(x) = \frac{x^6}{6!} + \frac{2x^7}{7!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} \quad (20)$$

and so on for other components. Using (15), the series solution is therefore given by

$$v(x) = x + \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \dots \dots \dots\right) \quad (21)$$

that converges to the exact solution

$$v(x) = x + e^x \quad (22)$$

B. APPLICATION:2

Consider Linear Volterra integro-differential equations of second kind

$$v''(x) = 1 + x + \int_0^x (x-t)v(t)dt \text{ with } v(0) = 1, v'(0) = 1 \quad (23)$$

Applying the Laplace transform to both sides of (23) and using initial conditions, we get

$$L\{v(x)\} = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} + \frac{1}{p^2}L\{\int_0^x (x-t)v(t)dt\} \quad (24)$$

Using convolution theorem of the Laplace transform, we have

$$L\{v(x)\} = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} + \frac{1}{p^4} L\{v(x)\} \tag{25}$$

Operating inverse Laplace transform on both sides of (25), we have

$$v(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + L^{-1}\left\{\frac{1}{p^4} L\{v(x)\}\right\} \tag{26}$$

The Laplace decomposition algorithm assumes the solution v(x) can be expanded into infinite series as

$$v(x) = \sum_{n=0}^{\infty} v_n(x) \tag{27}$$

By substituting (27) in (26), the solution can be written as

$$\sum_{n=0}^{\infty} v_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + L^{-1}\left\{\frac{1}{p^4} L\left\{\sum_{n=0}^{\infty} v_n(x)\right\}\right\} \tag{28}$$

From (28), our required recursive relations are given by

$$v_0(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \tag{29}$$

$$v_{n+1}(x) = L^{-1}\left\{\frac{1}{p^4} L\{v_n(x)\}\right\}, n \geq 0 \tag{30}$$

The first few components of $v_n(x)$ by using recursive relation (18) as follows immediately

$$v_1(x) = \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} \tag{31}$$

$$v_2(x) = \frac{x^8}{8!} + \frac{x^9}{9!} + \frac{x^{10}}{10!} + \frac{x^{11}}{11!} \tag{32}$$

and so on for other components. Using (27), the series solution is therefore given by

$$v(x) = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots\ldots\ldots\ldots\right) \tag{33}$$

that converges to the exact solution

$$v(x) = e^x \tag{34}$$

APPLICATION : 3

Consider Linear Volterra integro-differential equations of second kind

$$v''(x) = 1 + x - 2x^2 + \int_0^x (x-t)v(t)dt \quad with\ v(0) = 5, v'(0) = 1, v''(0) = 1 \tag{35}$$

Applying the Laplace transform to both sides of (35) and using initial conditions, we get

$$L\{v(x)\} = \frac{5}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} - \frac{4}{p^6} + \frac{1}{p^3} L\{\int_0^x (x-t)v(t)dt\} \tag{36}$$

Using convolution theorem of the Laplace transform, we have

$$L\{v(x)\} = \frac{5}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} - \frac{4}{p^6} + \frac{1}{p^5} L\{v(x)\} \tag{37}$$

Operating inverse Laplace transform on both sides of (37), we have

$$v(x) = 5 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{4x^5}{5!} + L^{-1}\left\{\frac{1}{p^5} L\{v(x)\}\right\} \tag{38}$$

The Laplace decomposition algorithm assumes the solution v(x) can be expanded into infinite series as

$$v(x) = \sum_{n=0}^{\infty} v_n(x) \tag{39}$$

By substituting (39) in (38), the solution can be written as

$$\sum_{n=0}^{\infty} v_n(x) = 5 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{4x^5}{5!} + L^{-1}\left\{\frac{1}{p^5} L\left\{\sum_{n=0}^{\infty} v_n(x)\right\}\right\} \tag{40}$$

From (40), our required recursive relations are given by

$$v_0(x) = 5 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{4x^5}{5!} \tag{41}$$

$$v_{n+1}(x) = L^{-1}\left\{\frac{1}{p^5} L\{v_n(x)\}\right\}, n \geq 0 \tag{42}$$

The first few components of $v_n(x)$ by using recursive relation (40) as follows immediately

$$v_1(x) = \frac{5x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} - \frac{4x^{10}}{10!} \tag{43}$$

$$v_2(x) = \frac{5x^{10}}{10!} + \frac{x^{11}}{11!} + \frac{x^{12}}{12!} + \frac{x^{13}}{13!} + \frac{x^{14}}{14!} - \frac{4x^{15}}{15!} \tag{44}$$

and so on for other components. Using (39), the series solution is therefore given by

$$v(x) = 4 + \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots\ldots\ldots\ldots\right) \tag{45}$$

that converges to the exact solution

$$v(x)=4+e^x \tag{46}$$

IV CONCLUSIONS

In this paper, we have successfully developed the Laplace decomposition algorithm for the solution of linear Volterra integro-differential equations of second kind. The given applications showed that the exact solution have been obtained even with just first three terms of the Laplace decomposition algorithm solution, which indicates that the proposed method Laplace decomposition algorithm need much less computational work.

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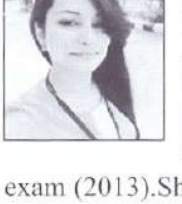

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